بِسمِ اللّهِ الرَّحمٰنِ الرَّحِيمِ

Devoir Surveillé $N°$ 6

$\int$ **Il sera tenu compte, dans l'appréciation des copies, de la précision des raisonnements ainsi que la clarté de la rédation.** y'

PCSI 1

Questions de Cours

Cours

Exercice 1

On pose $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin(t)}{\sin(t)+\cos(t)}dt$ et $J = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos(t)}{\sin(t)+\cos(t)}dt$

1. Using the change of the variable, $s = \frac{\pi}{2} - t$, we get
$$I = -\int_{\frac{\pi}{2}}^{0} \frac{\cos(s)}{\cos(s)+\sin(s)}ds = \int_0^{\frac{\pi}{2}} \frac{\cos(s)}{\sin(s)+\cos(s)}ds = J.$$
2. $I + J = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos(t)+\sin(t)}{\cos(t)+\sin(t)}dt = \int_0^{\frac{\pi}{2}} dt = \frac{\pi}{2}.$
3. We have $I = J$ and $I + J = \dfrac{\pi}{2}$, hence $I = J = \frac{\pi}{4}$.
4. The value of $\displaystyle\int_0^1 \frac{1}{t+\sqrt{1-t^2}}dt$:

   By change of the variable, $t = \sin(u)$, we get (note that $dt = \cos(u)du$), $\displaystyle\int_0^1 \frac{1}{t+\sqrt{1-t^2}}dt =$
$$\int_0^{\frac{\pi}{2}} \frac{\cos(u)}{\sin(u)+\sqrt{\cos^2(u)}du} = \int_0^{\frac{\pi}{2}} \frac{\cos(u)}{\sin(u)+\cos(u)}du = J = \frac{\pi}{4}.$$

Exercice 2

1. Determination of a primitive (not all primitives), of $t \mapsto \sin(2t)e^{\cos(t)}$.
$$\begin{aligned}\int_0^x \sin(2t)e^{\cos(t)}dt &= 2\int_0^x \cos(t)\sin(t)e^{\cos(t)}dt = -2\int_0^x \cos(t)\left(e^{\cos(t)}\right)' \\ &= \left[-2\cos(t)e^{\cos(t)}\right]_0^x - 2\int_0^x \sin(t)e^{\cos(t)}dt \\ &= \left[-2\cos(t)e^{\cos(t)}\right]_0^x + 2\int_0^x \left(e^{\cos(t)}\right)'dt = \left[-2\cos(t)e^{\cos(t)}\right]_0^x + \left[2e^{\cos(t)}\right]_0^x\end{aligned}$$

It follows that $x \mapsto -2\cos(x)e^{\cos(x)} + 2e^{\cos(x)} + 2e - 2$ is a primitive a primitive of $x \mapsto \sin(2x)e^{\cos(x)}$. In particular $x \mapsto -2\cos(x)e^{\cos(x)} + 2e^{\cos(x)}$ is also a primitive.

2. Résoudre l'équation différentielle suivante : $(E) \quad y' - \sin(t)y = \sin(2t)$:
Resolution of the homogeneous equation $(E_0) : y' - \sin(t)y = 0$:
The function $t \mapsto \cos(t)$ is a primitive of $-\sin$. Hence any solution y_H of (E_0) has the form

$$y_H : t \mapsto \lambda e^{-\cos(t)} \quad , \quad \lambda \in \mathbb{R}$$

Research of a particular solution (by variation of the constant) of the form $t \mapsto \lambda(t)e^{-\cos(t)}$. We replace in the equation (E), and we get $\lambda'(t)e^{-\cos(t)} = \sin(2t)$, so $\lambda'(t) = \sin(2t)e^{\cos(t)}$. hence the function λ is a primitive of $t \mapsto \sin(2t)e^{\cos(t)}$. By the previous question it is enough to take $\lambda(t) = -2\cos(t)e^{\cos(t)} + 2e^{\cos(t)}$. Thus $t \mapsto \left(-2\cos(t)e^{\cos(t)} + 2e^{\cos(t)}\right)e^{-\cos(t)} = -2\cos(t) + 2$ is a particular solution of (E). It follows that the solution of E has the form

$$t \mapsto \lambda e^{-\cos(t)} - 2\cos(t) + 2 \ , \ \lambda \in \mathbb{R}.$$

Exercice 3

Résoudre l'équation différentielle suivante : $(E) \quad y'' - 2y' + y = e^t$.
The characteristic equation is : $(\star) \quad x^2 - 2x + 1 = 0$ ($\delta = 0$ and $r = 1$).
The general solution of the homogeneous equation has the form $y_H : t \mapsto (\alpha t + \beta)e^t \quad , \quad \alpha, \beta \in \mathbb{R}$.
Since 1 is a solution of $(\star)$, and $\Delta = 0$, (E) has a particular solution of the form $t \mapsto at^2e^t$, after some calculus we get $a = \frac{1}{2}$. Thus the solutions of (E) are of the form $t \mapsto (\alpha t + \beta)e^t + \frac{1}{2}t^2e^t \ , \ \alpha, \beta \in \mathbb{R}$.

PROBLÈME

Formule de Taylor-Lagrange

Dans tout le problème, a, b désigne deux reels tels que $a < b$.

Première partie : Formule de la moyenne

Dans cette partie, $f, g : [a, b] \to \mathbb{R}$ sont deux fonctions continues sur $[a, b]$, avec g **positive sur** $[a, b]$.

1. Since g is continuous, positive and $\int_a^b g(t)dt = 0$, we get $g = 0$, hence $\int_a^b f(t)g(t)dt = 0$.
   On suppose dans la suite de cette partie que $\int_a^b g(t)dt \neq 0$.
2. f is a continuous function on $[a, b]$, hence it is bounded and attains its bounds.
3. Using the fact that g is positive, we get $f(\alpha)g(t) \leq f(t)g(t) \leq f(\beta)g(t)$, and by integration we obtain, $f(\alpha)\int_a^b g(t)dt \leq \int_a^b f(t)g(t)dt \leq f(\beta)\int_a^b g(t)dt$.

(4.) Note that $\int_a^b g(t)dt > 0$. The previous inequality give $f(\alpha) \leq \frac{\int_a^b f(t)g(t)dt}{\int_a^b g(t)dt} \leq f(\beta)$. Since f is continuous, by the Main Value Theorem (TVI), there exists $c \in [a,b]$ such that $f(c) = \frac{\int_a^b f(t)g(t)dt}{\int_a^b g(t)dt}$, that is $\int_a^b f(t)g(t)dt = f(c)\int_a^b g(t)dt$.

(5.) This is a particular case of the previous result, where $g = 1$.

Deuxième partie : Formule de Taylor

Dans cette partie, $f : [a,b] \to \mathbb{R}$ est une fonction de classe $\mathscr{C}^\infty$ sur $[a,b]$.

(6.) We have $\int_a^x f'(t)dt = f(x) - f(a)$, hence $f(x) = f(a) + \int_a^x f'(t)dt$.

(7.) By integration by parts, we have
$\int_a^x f'(t)dt = -\int_a^x (x-t)'f'(t)dt = -[(x-t)f'(t)]_a^x + \int_a^x (x-t)f''(t)dt = (x-a)f'(a) + \int_a^x (x-t)f''(t)dt$, hence $f(x) = f(a) + f'(a)(x-a) + \int_a^x (x-t)f''(t)dt$

(8.) By induction on n. Note that the formula is true for $n = 1$ (By the question 6.).
Now, assume that $f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)dt$. Looking the last term, we have

$$\begin{aligned}
\int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)dt &= -\int_a^x \left(\frac{(x-t)^n}{n!}\right)' f^{(n)}(t)dt \\
&= -\left[\frac{(x-t)^n}{n!} f^{(n)}\right] + \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt \\
&= \frac{(x-a)^n}{n!} f^{(n)}(a) + \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt
\end{aligned}$$

Hence

$$\begin{aligned}
f(x) &= \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)dt \\
&= \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{(x-a)^n}{n!} f^{(n)}(a) + \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt \\
&= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt
\end{aligned}$$

(9.) For $t \in [a,x]$, denote $g(t) = \frac{(x-t)^{n-1}}{(n-1)!}$. The two functions g and $f^{(n)}$ are continuous on $[a,x]$ and g positive. By the first part (question 4), there exists $c_x \in [a,x]$ such that $\int_a^x g(t)f^{(n)}(t)dt = f^{(n)}(c_x)\int_a^x g(t)dt = \frac{(x-a)^n}{n!} f^{(n)}(c_x)$. That is

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{(x-a)^n}{n!} f^{(n)}(c_x)$$

10. It is enough to prove that

$$\frac{(x-a)^n}{n!}f^{(n)}(c_x) =_a \frac{(x-a)^n}{n!}f^{(n)}(a) + o((x-a)^n)$$

In fact, we have $\frac{(x-a)^n}{n!}f^{(n)}(c_x) - \frac{(x-a)^n}{n!}f^{(n)}(a) = \frac{(x-a)^n}{n!}(f^{(n)}(c_x) - f^{(n)}(a))$. Since $a \leq c_n \leq x$, we get $\lim\limits_{x\to a} c_x = a$. On other hand $f^{(n)}$ is continuous (in particular on a), hence $\lim\limits_{x\to a} f^{(n)}(c_x) - f^{(n)}(a) = 0$. It follow that $f^{(n)}(c_x) - f^{(n)}(a)_a = o(1)$. So $\frac{(x-a)^n}{n!}f^{(n)}(c_x) - \frac{(x-a)^n}{n!}f^{(n)}(a) = \frac{(x-a)^n}{n!}o(1) = o((x-a)^n)$. We get the result as desired.

END