بِسمِ اللّهِ الرَّحمٰنِ الرَّحِيمِ

Devoir Surveillé N° 6

dim

Il sera tenu compte, dans l'appréciation des copies, de la précision des raisonnements ainsi que la clarté de la rédation.

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PCSI

Questions de Cours

Cours

Exercice 1

Soit E l'espace vectoriel $E=\mathbb{R}^3$, et considère les deux parties :

$$F=\{(x,y,z)\in\mathbb{R}^3\ /\ x-2y+z=0\}\quad \text{et}\quad G=\{(x,y,z)\in\mathbb{R}^3/\ 2x-y=0\ ,\ \ x-z=0\}$$

1. We have $0-2.0+0=0$ hence $(0,0,0)\in F$.
Let $u=(x,y,z), v=(x',y',z')\in F$ and $\lambda\in\mathbb{R}$. $u+\lambda v=(x+\lambda x', y+\lambda y', z+\lambda z')$. We have $(x+\lambda x')-2(y+\lambda y')+(z+\lambda z')=(x-2y+z)+\lambda(x'-2y'+z')=0$, hence $u+\lambda v\in F$.
2. We have $2.0-0=0$ and $0-0$ hence $(0,0,0)\in G$.
Let $u=(x,y,z), v=(x',y',z')\in G$ and $\lambda\in\mathbb{R}$. $u+\lambda v=(x+\lambda x', y+\lambda y', z+\lambda z')$. We have $2(x+\lambda x')-(y+\lambda y')=(2x-y)+\lambda(2x'-y')=0$ and $(x+\lambda x')-(z+\lambda z')=(x-z)+\lambda(x'-z')=0$, hence $u+\lambda v\in G$.
3. Let $u=(x,y,z)\in F$, hence $x=2y-z$, so $u=(2y-z,y,z)=y\underbrace{(2,1,0)}_{\in F}+z\underbrace{(-1,0,1)}_{\in F}$, so $F=\text{Vect}((2,1,0),(-1,0,1)=\text{Vect}((2,1,0),(1,0,-1))$ (use the fact that $\text{Vect}(u,v)=\text{Vect}(u,-v)$).
4. Let $u=(x,y,z)\in G$, hence $y=2x$ and $z=x$, so $u=(x,2x,x)=x\underbrace{(1,2,1)}_{\in G}$, so $G=\text{Vect}((1,2,1))$.
5. $((2,1,0),(1,0,-1))$ is a generated family of F, and it is free : if $a,b\in\mathbb{R}$ such that $a(2,1,0)+b(1,0,-1)=0$, then $2a+b=0$, $a=0$ and $-b=0$, that is $a=b=0$. It follows that $\dim F=2$.
The family $((1,2,1))$ is a basis of G hence $\dim G=1$.
6. Let $u\in F\cap G$. $u\in G$, then there exists $\alpha\in\mathbb{R}$ such that $u=\alpha(1,2,1)=(\alpha,2\alpha,\alpha)$. On the other hand $u\in F$ implies that $\alpha-2(2\alpha)+\alpha=0$, so $\alpha=0$. Thus $u=0$. But $\dim F+\dim G=2+1=3=\dim\mathbb{R}^3$. It follows that $F\oplus G=\mathbb{R}^3$, that if F and G are supplementary vector subspaces.

Exercice 2

Soit l'espace vectoriel $E=\mathbb{R}^4$, et notons (e_1,e_2,e_3,e_4) la base canonique de $\mathbb{R}^4$. Soit f l'unique endomorphisme de E vérifiant $f(e_1)=e_1+e_2$, $f(e_2)=f(e_3)=e_1$ et $f(e_4)=-e_1+e_4$.

1. $f(x,y,z,t)=(x+y+z-t,x,0,t)$.

2. Let $u=(x,y,z,t)\in\ker f$, then $x+y+z-t=0$, $x=0$ and $t=0$, hence $z=-y$, $x=t=0$, so $u=(0,y,-y,0)=y\underbrace{(0,1,-1,0)}_{\in\ker f}$. Thus $\ker f=\mathrm{Vect}((0,1,-1,0))$. In particular $\dim\ker f=1$.

3. By the rank formula $\mathrm{rg}(f)=\dim\mathbb{R}^4-\dim\ker f=4-1=3$.

PROBLÈME

Commutant d'un endomorphisme

Dans tout le problème E désigne un $\mathbb{K}$-espace vectoriel et $\mathscr{L}(E)$ l'espace vectoriel des endomorphismes de E.
Pour $f,g\in\mathscr{L}(E)$, on rappelle que fg désigne l'endomorphsime $f\circ g$. Pour $f\in\mathscr{L}(E)$, on rappelle que $f^0=\mathrm{Id}_E$ et pour $n\geq 1$, $f^n=\underbrace{f\circ f\circ\ldots\circ f}_{n\text{ fois}}$.
Pour $f\in\mathscr{L}(E)$, on note $\mathscr{C}(f)$ l'ensemble des endomorphismes de E qui commutent avec f c'est-à-dire

$$\mathscr{C}(f)=\{g\in\mathscr{L}(E)\ /\ fg=gf\}$$

Première partie :
Structure de $\mathscr{C}(f)$

Dans cette partie f est un endomorphisme de E.

1. We have $\mathrm{Id}_E f=f=f\,\mathrm{Id}_E$, hence $\mathrm{Id}_E\in\mathscr{C}(f)$.

2. We have $0f=0=f0$, hence $0\in\mathscr{C}(f)$.
Let $g,h\in\mathscr{C}(f)$ and $\lambda\in\mathbb{K}$. Note that $gf=fg$ and $hf=fh$, hence $(g+\lambda h)f=gf+\lambda hf=fg+\lambda fh=f(g+\lambda h)$, so $f+\lambda h\in\mathscr{C}(f)$. Thus $\mathscr{C}(f)$ is a vector subspace of $\mathscr{L}(E)$.

3. If $g,h\in\mathscr{C}(f)$, then $ghf=gfh=fgh$, hence $gh\in\mathscr{C}(f)$.

4. By induction on $n\in\mathbb{N}$.
For $n=0$, $g^0=\mathrm{Id}_E\in\mathscr{C}(f)$.
Assume that $g^n\in\mathscr{C}(f)$ for some $n\in\mathbb{N}$, since $g\in\mathscr{C}(f)$ and $g^n\in\mathscr{C}(f)$, by the previous question we get $g^ng\in\mathscr{C}(f)$, that is $g^{n+1}\in\mathscr{C}(f)$.

5. Since $\mathscr{C}(f)$ is a vector subspace and each element f^k is in $\mathscr{C}(f)$, it follows that $a_0\mathrm{Id}_E+a_1f+\ldots+a_nf^n\in\mathscr{C}(f)$.

Deuxième partie :
Cas d'un endomorphisme cyclique

Dans toute la **suite du problème** E est un espace vectoriel de **dimension** 3 ($\dim E=3$).
On suppose dans **cette partie** qu'il existe $v\in E$ tel que la famille $(v,f(v),f^2(v))$ est **libre**.

6. $(v,f(v),f^2(v))$ is a free familly with $3(=\dim E)$ elements, so it is a basis of E.

7. Soient $g,h\in\mathscr{C}(f)$ tels que $g(v)=h(v)$.

   7.1 $g(f(v))=(gf)(v)=(fg)(v)=f(g(v))=f(h(v))=(fh)(v)=(hf)(v)=h(f(v))$.

   7.2 By the same argument as in the previous question
   $g(f^2(v))=(gf^2)(v)=(f^2g)(v)=f^2(g(v))=f^2(h(v))=(f^2h)(v)=(hf^2)(v)=h(f^2(v))$.

   7.3 Let $x\in E$, since $(v,f(v),f^2(v))$ is a basis of E, there exists $a,b,c\in\mathbb{K}$ such that $x=av+bf(v)+cf^2(v)$, hence $g(x)=g(av+bf(v)+cf^2(v))=ag(v)+bg(f(v))+cg(f^2(v))=ah(v)+bh(f(v))+ch(f^2(v))=h(av+bf(v)+cf^2(v))=h(x)$. It follows that $g=h$.

8. Soit $g\in\mathscr{C}(f)$.

$\boxed{8.1}$ Since $(v, f(v), f^2(v))$ is a basis of E and $g(v) \in E$, there exists $\alpha, \beta, \gamma \in \mathbb{K}$ such that $g(v) = \alpha v + \beta f(v) + \gamma f^2(v)$.

$\boxed{8.2}$ Denote $h = \alpha \mathrm{Id}_E + \beta f + \gamma f^2$. By the the result of the question 5., $h \in \mathscr{C}(f)$. On the other hand $h(v) = (\alpha \mathrm{Id}_E + \beta f + \gamma f^2)(v) = \alpha v + \beta f(v) + \gamma f^2(v) = g(v)$. It follows that (by the result of the question 7) $g = h = \alpha \mathrm{Id}_E + \beta f + \gamma f^2$.

$\boxed{9.}$ If $g \in \mathscr{C}(f)$, by the previous question there exists $\alpha, \beta, \gamma \in \mathbb{K}$ such that $g = \alpha \mathrm{Id}_E + \beta f + \gamma f^2$, hence $g \in \mathrm{Vect}(\mathrm{Id}_E, f, f^2)$. On the other hand, since $\mathrm{Id}_E, f, f^2 \in \mathscr{C}(f)$, we get $\mathrm{Vect}(\mathrm{Id}_E, f, f^2) \subseteq \mathscr{C}(f)$. Thus $\mathscr{C}(f) = \mathrm{Vect}(\mathrm{Id}_E, f, f^2)$.

$\boxed{10.}$ By the previous question (Id_E, f, f^2) is a generated family of Vcet. Let $a, b, c \in \mathbb{K}$ such that $a\mathrm{Id}_E + bf + cf^2 = 0$, in particular $(a\mathrm{Id}_E + bf + cf^2)(v) = 0$, that is $av + bf(v) + cf^2(v) = 0$, hence $a = b = c = 0$ (since $(v, f(v), f^2(v))$ is free). It follows that (Id_E, f, f^2) is a basis of $\mathscr{C}(f)$, so $\dim \mathscr{C}(f) = 3$.

$\boxed{11.}$ We have $f^3 \in \mathscr{C}(f) = \mathrm{Vect}(\mathrm{Id}_E, f, f^2)$, hence there exists $a, b, c \in \mathbb{K}$ such that $f^3 = a\mathrm{Id}_E + bf + cf^2$.

Troisième partie :
Cas d'un endomorphisme nilpotent

Dans cette partie f est un endomorphisme nilpotent de E d'indice de nilpotence égale à 3 c'est-à-dire $f^3 = 0$ et $f^2 \neq 0$.

$\boxed{12.}$ Follows from the fact that $f^2 \neq 0$.

$\boxed{13.}$ Let $a, b, c \in \mathbb{K}$, such that $av + bf(v) + cf^2(v) = 0 \ (\star)$. Applying f^2 to $(\star)$, we get $af^2(v) + bf^3(v) + cf^4(v) = 0$, so $af^2(v) = 0$, hence $a = 0$ (since $f^2(v) \neq 0$). Now $(\star)$ become $bf(v) + cf^2(v) = 0$. Applying f to $(\star)$ and this yields $bf^2(v) = 0$, it follows that $b = 0$. Finally we get $af(v) = 0$ (from $(\star)$), so $af^2(v) = 0$, hence $a = 0$. Thus the family $(v, f(v), f^2(v))$ is free.

$\boxed{14.}$ There exists $v \in E$ such that $(v, f(v), f^2(v))$ is a basis of E, hence free. By the result of the second part, we get $\mathscr{C}(f) = \mathrm{Vect}(\mathrm{Id}_E, f, f^2)$.

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